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Logica Universalis

Logic, Spatial Algorithms and Visual Reasoning

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Abstract. Spatial and diagrammatic reasoning is a significant part not only of logical abilities, but also of logical studies. The authors of this paper consider some novel trends in studying this type of reasoning. They show that there are the following two main trends in spatial logic: (i) logical studies of the distribution of various objects in space (logic of geometry, logic of colors, etc.); (ii) logical studies of the space algorithms applied by nature itself (logic of swarms, logic of fungi colonies, etc.).

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1. Introduction

How important spatial visualization ability is for us humans, we know from our everyday life as well as from all areas of science: We do not only move and orientate ourselves in space, but we also examine spatial objects with our logical abilities or use spatial relations to examine our logical abilities with them. This close connection of spatial and logical capabilities is not only reflected in geometry or diagrammatic logic, but also in everyday metaphors, mental images and visualisations that we need to find our way in space or to use the space for our creative processes. We now know from fields such as cognitive sciences, animal learning, and evolutionary psychology that not only humans think in spatial relations, but animals in general [5]. Furthermore, we also know that not only our understanding of animal intelligence, but also machine or artificial intelligence depends on or at least benefits from our knowledge of the connection between space and logic [21].

Before we will give an overview of the papers in this Special Issue in Sect. 4, we would like to discuss some advantages and disadvantages of visual reasoning in Sect. 2 and give a brief history of the topics covered here in Sect. 3.

2. Advantages and Disadvantages of Visual Reasoning

Although there have always been periods in history when visual and spatial reasoning were used intensively, there are some prejudices against diagrams, figures, maps or visual reasoning in general, which were especially popular in the late nineteenth and early twentieth centuries [7]: First, there are 'misleading intuitions' when visual reasoning can lead to understanding and invention, but there is no corrective through the certainty of logical proof. In this case, visual reasoning can be a false guide. Second, there is the problem of the 'particularity of intuitive evidence'. A classic example of this is the visual proofs of the Pythagorean theorem, which often visualise the corresponding Euclidean theorem (*Elem.* I 49), but not all possible cases of it. Third, there are problems entitled 'epistemic vacuity'. Visual reasoning may confirm what we already know on an intuitive basis, but epistemologically they add little to our knowledge of truth. Visual-spatial reasoning can thus symbolise facts in logic or mathematics, but not reasons.

This phase of the late nineteenth and early twentieth century, often referred to as the 'crisis of intuition', came out of mathematics and physics and made the aforementioned arguments against visual reasoning strong: Intuition was considered unscientific, whereas rigour and exactness became buzzwords in mathematics and logic [17]. The twentieth century crisis of intuition began to wane from the 1960s onward and in the 1990s started a trend often referred to as the 'spatial' or 'visual turn', motivated in particular by developments in logic, mathematics, psychology and AI. This development not only corresponds to the perception of those involved in visual reasoning in the field of logic and mathematics, but it is also evidenced, for example, by empirical studies that have examined the frequency of certain diagrams in mathematical journals [16].

A milestone in the study of visual-spatial communication in logic and mathematics was Martin Gardner's *Logic Machines and Diagrams* [9], which showed that diagrams in combinatorics and logic could be transferred to technical forms such as the first logic machines and computers. In the 1980s, the graphical user interfaces of the first personal computers motivated the development of the first research groupes and institutes in America (e.g. 'Visual Inference Labaratory' at lndiana University) and Europe (e.g. 'Philosophische Diagrammatik' at FernUniversität in Hagen) [3].

The breakthrough in philosophy and mathematics, however, was Sun-Joo Shin's paradigmatic work *The Logical Status of Diagrams* in 1993, in which Shin showed that visual communication can be used in logic on a par with other algebraic languages [26]: She was able to successfully demonstrate that diagrams, and thus any form of visual-spatial reasoning in logic, have their own syntax and semantics, allowing properties of formal systems to be proved.

While many researchers continued to develop this approach in logic, others were able to transfer this approach to mathematics a few years later and thus simultaneously created the foundations of an application of visual communication in the AI field [15]. Atsushi Shimojima's paradigmatic research for cognitive science [25] showed that diagrams can not only be used as a formal system, but that they have cognitive advantages that algebraic and arithmetic notations do not possess. Nowadays, numerous advantages of diagrams are known, with buzzwords such as free rides, observational advantages, operational iconicity, etc. and currently used in logic, mathematics and related subjects.

One can continue to argue about the advantages and disadvantages of visual reasoning in logic and mathematics, but every form of representation has such advantages and disadvantages, even the linear representation of the formulas once labelled as strict and exact. Willard Van Orman Quine (1908–2000), for example, saw certain advantages in using so-called 'quantificational diagrams' over linear logical notation, but rejected the idea because the diagrams were too cumbersome [2]. Even if the three disadvantages mentioned above, 'misleading intuitions', 'particularity of intuitive evidence' and 'epistemic vacuity', are not eliminated, the advantages of diagrams and maps are already familiar to us from everyday life: No one would prefer a verbal description or linear formalisation to a map when one wants to get an orientation in the railway network.

3. Some Glimpses into the History of Visual Reasoning

The matter is that spatial visualizations are an important ability in logical and mathematical thinking as such. As important as this awareness was again at the end of the 20th century, this trend is by no means new in research. Among phytagoreans, for example, we find a strong understanding that visual reasoning is profitable in both logic and mathematics [11,23]. That Aristotle used logic diagrams is hardly doubted by any researcher, although such diagrams have survived only from his direct students [20]. Moreover, several authors report the use of logic diagrams in didactics of antiquity (e.g. Aug. Conf. IV 16).

The oldest logic diagrams preserved today in original manuscripts are tree diagrams, logical squares and pons asinorum diagrams [14,29]. Take the Square of Opposition for example. So, for the first time, it was introduced in the *Peri Hermeneias*, the guide to Aristotelian logic that was attributed (perhaps wrongly) to Lucius Apuleius of Madaura (ca. 124–ca. 170). The author of this book noted that the four propositions are related within the framework of 'squared figure' (*quadrata formula*), where the dedicative and abdicative universals are located on the top line (*in superiore linea*) and the dedicative and abdicative particulars are on the bottom line (*in inferiore linea*) [19, p. 110].

Later a lot of various visualizations of logical notions have appeared. But one of the revolutionary techniques should be especially mentioned. It is Euler diagrams. This technique was probably already known by the twelfth century Baghdad scholar Abu'l-Barakāt al-Baghdādī and was rediscovered several times in the early modern period [18]. With Barakāt, one can also see how closely the diagrams are related to the understanding of an algorithm [12]. These diagrams then became known in the eighteenth and nineteenth century through Immanuel Kant (1724–1804), who in turn had adopted them from Euler. In the letters XXXIV–XXXVII (1761) from the *Lettres à une princesse d'Allemagne sur divers sujets de physique et de philosophie* [8, pp. 259–273] written by Leonhard Euler (1707–1783) we find the visualization of three main types of the set-theoretic relationships (inclusion, exclusion, intersection) by means of circles in a two-dimensional plane, where each circle means a set such that (i) the first circle containing another one within the interior means the inclusion of the second set, (ii) the two overlapping circles represent two intersecting sets, (iii) the two circles without joint elements represent two disjoint sets. This technique was developed then by John Venn (1834–1923), whose technique is now called Venn diagrams [28].

Another example of fruitful visualization in logical thinking is presented by abstract machines, first invented by Alan Mathison Turing (1912–1954), who has introduced a visualization of decidable arithmetic functions. In turn, these abstract machines were a continuation of old logical tradition of diagrammatic reasoning. This tradition was established in the medieval magic literature of Arabic, Hebrew, Latin, and Hindu authors from the twelfth to the thirteenth century, who have described some numerical tools as a kind of logical machines. Hence, the diagrammatic reasoning implemented later in Turing machines has a rich history and pre-history.

Let us remember that this pre-history starts with different numerical squares such as magic squares and Vedic squares, considered in the book *De Occulta Philosophia* written by Henry Cornelius Agrippa von Nettesheim (1486–1535). It became later very popular in Europe. But this book was not first on this subject. Much earlier the *Kitāb Šams al-Ma'ārif wa Laṭāif al-'Awārif* (كتاب شمس المعارف ولطائف العوارف, *Book of the Sun of Gnosis and the Subtleties of Elevated Things*), devoted to numerical tools in magic, was written by Aḥmad ibn 'Alī ibn Yūsuf al-Būnī (ca. 1145–1225). This type of research was connected to Arabic astronomical calculations such as the *Tables of Toledo*, giving the seven mathematical squares in relation to the virtues of the seven known planets. One of the authors of these tables was Ibrāhīm ibn Yaḥyā al-Zarqālī (1029–1087), who lived in Toledo. Then this tradition influenced some Hebrew and Latin authors such as Rabbi 'Aḇrāhām ben Mē'īr 'Ībn 'Ēzrā' (Arabic: Ibrāhīm al-Majīd ibn 'Azrā; 1092–1167) and Alfonso X of Castille (1221–1284).

In course of time the numerical squares and other magic numerical tools of Arabic, Hebrew, and Latin authors from the twelfth century to the thirteenth century influenced Ramon Llull (ca. 1232–ca. 1315) in his work on logical machines. So, in his *Ars generalis ultima* or *Ars magna* (*Ultimate General Art* or *Great Art*), published in 1305, he proposed a kind of logical machine [4] that was well formally explicated much later by Gottfried Wilhelm von Leibniz

(1646–1716) as the famous *characteristica universalis* (universal characteristic; [24]). Quite close ideas of logical machines based on letter calculations are mentioned in the works by Abū Zayd ʻAbd ar-Raḥmān ibn Muḥammad ibn Khaldūn al-Ḥaḍramī (1332–1406), see [13]. The *characteristica universalis* is called there *zāyirja* (زايرجة). The same ideas can be observed in the works of Hindu authors, too.

Hence, the roots of the Leibnizian idea of *characteristica universalis* as the pre-history of Turing machines can be detected in the texts of different Medieval authors: Latin, Hebrew, Arabic, and even Hindu.

By now, various visualizations of logical notions are considered especially helpful in logical studies. First, they may be used in teaching some basic logical courses, as it was made by Lewis Carroll (Charles Lutwidge Dodgson; 1832–1898) for syllogistic in his *The Game of Logic* published in [6] (see [22]). Second, they can allow us to make some significant inventions in logic. So, Friedrich Ludwig Gottlob Frege (1848–1925) in his *Begriffsschrift* [10] has defined connectives and quantifiers as different lines connecting atomic formulas in order to introduce the first axiomatization of the first-order logic. Peter Henry George Aczel (born 1941) has used a visual (graph) representation for defining sets and in this way has formulated his anti-foundation axiom that every graph has a unique decoration in order to introduce non-well-founded set theory [1]. In this theory, there are non-well-founded sets if they are pictured by the graph containing at least one infinite path. Yisrael Ury [27] has mathematically formalized the logical rule of *qal wa-ḥomer* from Judaic hermeneutics using logical diagrams.

4. Content of the Special Issue

There are three main approaches to diagrammatic reasoning we would like to present in our special issue: logical, mathematical, and engineering. In the first (logical) approach, various diagrams for explicating logical notions and relations are studied. In the mathematical approach, different spatial notions are studied from the point of view of geometry or topology. In the engineering approach, different natural processes are regarded as specific abstract machines. The latter direction of investigation is called unconventional computing. Here any natural process is regarded as a kind of visual computation, such as DNA computing, reaction-diffusion computing, social insects computing, etc. As a result, nature itself is considered a visual demonstration of computations that we can learn.

Thus, this special issue of *Logica Universalis* is devoted to diagrammatic reasoning in general and visual demonstrations of computations and logical thinking. The aim is to bring together logicians or philosophers dealing with diagrammatic tools with researchers dealing with explicating processes in human actions and in nature itself by means of considering biological, bio-inspired, chemical, physical, etc. computing to initiate development of novel diagrammatic paradigms.

The special issue consists of seven contributions. In the paper *Two Squares of Opposition in Two Arabic Treatises: al-Suhrawardī and al-Sanūsī* opening the special issue, Saloua Chatti analyzes two squares of opposition which occur in the logical treatises of Shihāb al-Dīn al-Suhrawardī (1155–1191) and Muhammed b. Yūsuf al-Sanūsī (1428–1490) in order to show that these squares differ from each other and from the traditional square introduced in the *Peri Hermeneias* attributed to Lucius Apuleius.

It is well known that the first explicit occurrence of axiomatic method in mathematical reasoning may be found in the *Elements* written by Euclid of Alexandria (ca. 325–270 B.C.). As a result, since antiquity the school geometry course based on the *Elements* have assumed some strong logical abilities to draw correct conclusions proving geometric theorems. In the paper *On the Logical Geometry of Geometric Angles*, Hans Smessaert and Lorenz Demey demonstrate that many geometric theorems may be proven thanks to using the logical relations among angles in the Euclidian plane geometry. They start with the basic tripartition of all angels into acute, right, and obtuse ones. Then they show a straightforward account of the Aristotelian syllogistic relations between angular concepts.

In the paper *Three-Dimensional Affine Spatial Logics*, Adam Trybus is concentrated on the three-dimensional case of spatial logic in order to demonstrate that it is possible to construct formulas describing a three-dimensional coordinate frame with addition and multiplication. Then he has proved that every region satisfies an affine complete formula, meaning that all regions satisfying it are affine equivalent.

The paper *Things May Not Be Simple: On Wittgenstein's Internal Relations* by Fabien Schang is devoted to logical atomism introduced in the *Tractatus Logico-Philosophicus* by Ludwig Josef Johann Wittgenstein (1889–1951). Fabien Schang considers logic of colors in order to show some problems in finding primary objects (logical atoms) for colors. The atoms are defined as the following properties (not objects): additive primary colors (red, blue, green) and subtractive primary colors (yellow, cyan, magenta). Then their logical combinations give different states of affairs including impossible colors (counterparts of incompatible propositions) and complementary colors (counterparts of contradictories). In this way logic of colors is constructed by different logical compositions of logical primary properties.

The next three papers are devoted to bio-inspired unconventional computing, that is, to representing different living creatures as a visual type of computation machines.

In the paper *Observation of Autonomous Behavioral Selection in Physarum Plasmodium* submitted by Tomohiro Shirakawa, Hiroshi Sato and Kazuki Ishimaru, the authors continue their studies presenting the behavior of plasmodia *Physarum polycephalum* as a kind of computers, but now they focus on the experiments, when a plasmodium changes its own behavior rules. The authors conclude that this phenomenon is bio-computationally important, because it shows some biological characteristics in computing which are close to the metaphysic notion of free will.

In the paper *Logics in Fungal Mycelium Networks* by Andrew Adamatzky, Phil Ayres, Alexander E. Beasley, Nic Roberts, Martin Tegelaar, Michail-Antisthenis Tsompanas and Han A. B. Wösten, some Boolean functions (gates) are defined within the framework of living mycelium networks. Hence, in these networks of fungi colonies they detect a form of distributed decision making based on Boolean logic. The mycelium colonies implement logic visually via propagation of electrical and chemical signals in pairs with morphological changes in the mycelium structure. This research is based on experiments with a single real colony of *Aspergillus niger* that was thus presented as a kind of computer.

In the paper *Brain and Its Universal Logical Model of Multi-Agent Biological Systems* by Jerzy Król, Andrew Schumann and Krzysztof Bielas, a topological model is defined how living agents (ants, bees, active particles of *Physarum polycephalum* and *Amoeba proteus*, etc.) concurrently react to a set of external stimuli located variously and with different powers of intensity. This model is generalized to the infinite covers with intuitionistic logic in order to introduce a category for the swarm orientation in space. This category is defined as a sheaf category.

Hence, there are the two main trends in visual reasoning, represented in this special issue: (i) logical studies of the distribution of various objects in space (logic of geometry, logic of colors, etc.); (ii) logical studies of the space algorithms applied by nature itself (logic of swarms, logic of fungi colonies, etc.). The two types of studies demonstrate some deep differences between spatial algorithms in inanimate nature and in living creatures. While in inanimate nature we can appeal to classical logic with clearly separated properties of the objects under study, in living beings, even at the level of unicellular organisms, we find phenomena of free will. Therefore some logical functions may be only approximated in their behavior and are not completely embodied. Spatial reasoning from inanimate to living nature becomes more complicated, revealing itself for itself, as Friedrich Wilhelm Joseph Schelling (1775–1854) and Arthur Schopenhauer (1788–1860) said in their *Naturphilosophie* (nature philosophy).

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